A NOTE ON THE UNIQUENESS OF INVOLUTION IN LOCALLY C*-ALGEBRAS

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ABSTRACT. In the present note we show that the involution in locally C*-algebras is uniquely determined.

1. Introduction

One of the important basic facts of the theory of C^***-algebras** is that the unary operation of involution in a C^*-algebra is uniquely determined. This property was first observed in 1955 by Bohnenblust and Karlin in [2] (see as well [7] for a nice exposition).

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [6]. The Hausdorff projective limits of projective families of C^*-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [3]. We will follow Inoue [4] in the usage of the name **locally** C^***-algebras** for these algebras.

The purpose of the present notes is to show that the unary operation of involution in locally C^*-algebras is uniquely determined.

2. Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutive algebra) is an algebra A over $\mathbb{C}$ with an involution

$$^* : A \to A,$$

such that

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

A seminorm $\|.\|$ on a *-algebra A is a C^*-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \|b\|,$$

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and satisfies the C^*-condition, i.e.

$$\|a^*a\| = \|a\|^2,$$

for every $a, b \in A$. Note that the C^*-condition alone implies that $\|.\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [3]).

When a seminorm $\|.\|$ on a *-algebra A is a C^*-norm, and A is complete in in the topology generated by this norm, A is called a **C^*-algebra**. The following theorem is valid.

Theorem 1 (Bohnenblust and Karlin [2]). *The unary operation of involution in a C^*-algebra is uniquely determined.*

Proof. See for example [7] for details. □

A topological *-algebra is a *-algebra A equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra A, one puts $N(A)$ for the set of continuous C^*-seminorms on A. One can see that $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|.\|_\alpha, \|.\|_\beta\} \in N(A)$$

for every $\|.\|_\alpha, \|.\|_\beta \in N(A)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological *-algebra A, and $\|.\|_\alpha \in N(A)$, $\alpha \in \Lambda$,

$$\ker \|.\|_\alpha = \{a \in A : \|a\|_\alpha = 0\}$$

is a *-ideal in A, and $\|.\|_\alpha$ induces a C^*-norm (we as well denote it by $\|.\|_\alpha$) on the quotient $A_\alpha = A/\ker\|.\|_\alpha$, and A_α is automatically complete in the topology generated by the norm $\|.\|_\alpha$, thus is a C^*-algebra (see [3] for details). Each pair $\|.\|_\alpha, \|.\|_\beta \in N(A)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism

$$g_\alpha^\beta : A_\beta \to A_\alpha.$$

Let, again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " $\preceq$ ". Let

$$\{A_\alpha, \alpha \in \Lambda\}$$

be a family of C^*-algebras, and g_α^β be, for

$$\alpha \preceq \beta,$$

the continuous linear *-mappings

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,$$

so that

$$g_\alpha^\alpha(x_\alpha) = x_\alpha,$$

for all $\alpha \in \Lambda$, and

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma,$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let A be a *-subalgebra of the direct product algebra

$$\prod_{\alpha\in\Lambda} A_\alpha,$$

so that for its elements

$$x_\alpha = g_\alpha^\beta(x_\beta),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_\alpha \in A_\alpha,$$

and

$$x_\beta \in A_\beta.$$

Definition 1. *The *-algebra A constructed above is called a **Hausdorff projective limit** of the projective family*

$$\{A_\alpha, \alpha \in \Lambda\},$$

relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\varprojlim A_\alpha,$$

and is called the Arens-Michael decomposition of A.

It is well known (see, for example [8]) that for each $x \in A$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : A \longrightarrow A_\beta,$$

defined by

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

Definition 2. *A topological *-algebra A over $\mathbb{C}$ is called a **locally C^*-algebra** if there exists a projective family of C^*-algebras*

$$\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$A \cong \varprojlim A_\alpha,$$

i.e. A is topologically *-isomorphic to a projective limit of a projective family of C^*-algebras, i.e. there exits its Arens-Michael decomposition of A composed entirely of C^*-algebras.

A topological *-algebra A over $\mathbb{C}$ is a locally C^*-algebra iff A is a complete Hausdorff topological *-algebra in which topology is generated by a saturated separating family of C^*-seminorms (see [3] for details).

Example 1. *Every C^*-algebra is a locally C^*-algebra.*

Example 2. *A closed *-subalgebra of a locally C^*-algebra is a locally C^*-algebra.*

Example 3. *The product $\prod_{\alpha\in\Lambda} A_\alpha$ of C^*-algebras A_α, with the product topology, is a locally C^*-algebra.*

Example 4. *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally C^*-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see* [5] *for details).*

Let A be a locally C^*-algebra. Then an element $a \in A$ is called **bounded**, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|.\|_\alpha \in N(A)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is well-known that for each locally C^*-algebra A, its set $b(A)$ of bounded elements of A is a locally C^*-subalgebra, which is a C^*-algebra in the norm $\|.\|_\infty$, such that it is dense in A in its topology (see for example [3]).

3. The uniqueness of involuton in locally C*-algebras

Here we present the main theorem of the current notes.

Theorem 2. *The unary operation of involution in any locally C^*-algebra is unique, i.e., if $(A, ^*, \|.\|_\alpha, \alpha \in \Lambda)$ and $(A, ^\#, \|.\|_\alpha, \alpha \in \Lambda)$ are two locally C^*-algebras, means that each seminorm $\|.\|_\alpha, \alpha \in \Lambda$, satisfies the C^*-property for both operations, " $*$ " and "$\#$", then*

$$^* = \#$$

on A.

Proof. Let now A be a locally C^*-algebra, and let

$$A = \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition, built using the family of seminorms $\|.\|_\alpha, \alpha \in \Lambda$, so that for each $\alpha \in \Lambda$,

$$(A_\alpha, *_\alpha, \|.\|_\alpha)$$

and

$$(A_\alpha, \#_\alpha, \|.\|_\alpha)$$

are C^*-algebras, where the unary operations " $*_\alpha$ " and "$\#_\alpha$" on A_α are defined as follows:

$$\pi_\alpha(x^*) = (\pi_\alpha(x))^{*_\alpha},$$

and

$$\pi_\alpha(x^\#) = (\pi_\alpha(x))^{\#_\alpha},$$

for each $x \in A$ and $\alpha \in \Lambda$.

Let us now assume, to the contrary to the statement of the theorem, that there exists some $x \in A$, such that

$$x^* = y \neq z = x^\#.$$

Then there must exist $\alpha_0 \in \Lambda$, such that

$$\pi_{\alpha_0}(y) \neq \pi_{\alpha_0}(z).$$

In fact, if it is not the case, and

$$\pi_\alpha(y) = \pi_\alpha(z)$$

for each $\alpha \in \Lambda$, implies that
$$y = z,$$
which contradicts the assumption.

So, α_0 must be such that
$$\pi_{\alpha_0}(x^*) \neq \pi_{\alpha_0}(x^{\#}),$$
which means that for
$$\pi_{\alpha_0}(x) = x_{\alpha_0} \in A_{\alpha_0},$$
$$x_{\alpha_0}^{*_\alpha} \neq x_{\alpha_0}^{\#_\alpha},$$
which contradicts Theorem 1. Found contradiction proves the theorem. □

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